Kondo physics in the algebraic spin liquid

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Abstract
We study Kondo physics in the algebraic spin liquid, a term recently proposed to describe
$ZnCu_3(OH)_6Cl_2$ (Ran *et al* 2007 *Phys. Rev. Lett.* **98** 117205). Although the spin dynamics of
the algebraic spin liquid is described by massless Dirac fermions, this problem differs from the
pseudogap Kondo model because the bulk physics in the algebraic spin liquid is governed by an
interacting fixed point where well-defined quasiparticle excitations are not allowed.
Considering an effective bulk model characterized by an anomalous critical exponent, we derive
an effective impurity action in the slave-boson context. Performing a large-N_σ analysis with a
spin index N_σ, we find an impurity quantum phase transition from a decoupled local-moment
state to a Kondo-screened phase. We evaluate the impurity spin susceptibility and specific heat
coefficient at zero temperature, and find that such responses follow power-law dependences due
to the anomalous exponent of the algebraic spin liquid. Our main finding is that Wilson's ratio
for the magnetic impurity depends strongly on the critical exponent in the zero temperature
limit. We propose that Wilson's ratio for the magnetic impurity may be one possible probe to
reveal the criticality of the bulk system.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recent experiments have claimed the emergence of spin
liquid (SL) phases in materials of geometrically frustrated
lattices such as Cs_2CuCl_4 [1], κ-$(ET)_2Cu_2(CN)_3$ [2], and
$ZnCu_3(OH)_6Cl_2$ [3], where no symmetries associated with
spin rotations (magnetic ordering) and lattice translations
(valance bond ordering) are broken at low temperatures while
charge fluctuations are frozen due to strong electron–electron
interactions (Mott insulator). An important issue is the nature
of such SL phases. Although spin susceptibility, specific heat,
and thermal transport measurements can determine possible
spin liquids, there still remains uncertainty.

Consider Cs_2CuCl_4 with an anisotropic triangular lat-
tice [1]. Although this material exhibits magnetic long-range
spiral ordering below $T = 0.62$ K with an incommensurate
wavevector, the spin-fluctuation spectrum in inelastic neutron
scattering experiments has shown a large high-energy contin-
uum beyond the spin-wave description. In addition, this contin-
uum spectrum survives above the Neel temperature. More de-
tailed analysis revealed that the continuum follows $\text{Im } \chi(\omega) \sim$
$\omega^{-\eta}$ with an anomalous exponent η, suggesting the presence
of deconfined critical spinons. Such spin-fluctuation measure-
ments suggest several candidates of SL scenarios, for exam-
ple, decoupled one-dimensional chains [4], proximate gapped

SLs [5], algebraic spin liquids (ASLs) [6], algebraic vortex liq-
uids [7], and so on [8].

Recently, Florens *et al* studied the role of magnetic
impurities in both the Z_2 SL phase and the O(4) quantum
critical point (QCP) separating the spiral magnetic order from
the Z_2 SL [9]. Although impurity moments coupled to
spin-1 bosons (spin singlet–triplet excitations) in conventional
paramagnets are only partially screened even at the bulk O(3)
QCP [10], they have shown that the presence of deconfined
bosonic spinons can display a bosonic version of the Kondo
effect. Furthermore, they found a weak-coupling impurity
quantum phase transition (I-QPT) from a local-moment state
to a fully screened phase. This study implies that the magnetic
impurity can be utilized as a probe for elementary excitations,
thus identifying the nature of SLs.

In this paper we investigate the Kondo effect in the
ASL, recently proposed to be realized in the Kagome
antiferromagnet $ZnCu_3(OH)_6Cl_2$ [11], where no magnetic
order is observed down to very low temperatures of about
50 mK compared to the Curie–Weiss temperature (>200 K),
and there is no sign of a spin gap in dynamical neutron
scattering [3]. However, it is not perfectly clear whether all
experiments are consistent with the ASL conjecture. The ASL
picture is not consistent with the temperature-linear specific
heat below 0.5 K and saturation of the spin susceptibility to
a finite value below 0.3 K [11], because these measurements

indicate the existence of a finite density of states at the Fermi energy. This discrepancy may result from the presence of disorder in real materials. To examine the role of magnetic impurities in the ASL can be an important test in revealing the genuine nature of the SL phase of this compound.

The ASL can be found from the fermion representation of the Heisenberg model via the flux mean-field ansatz [12]. This reminds us of the previous study of the Kondo effect in the flux phase by Cassanello and Fradkin [13]. More generally, one may regard the present impurity problem as the class of the pseudogap Kondo model [13–15], where the fermion density of states vanishes as $\rho(\epsilon) \sim |\epsilon|^r$ near the Fermi energy. The case of $r = 0$ corresponds to a Fermi liquid while the $r = 1$ case coincides with Dirac fermions arising from the flux phase or d-wave superconductor. In contrast with the Kondo effect of the Fermi liquid, the pseudogap Kondo model has shown that Kondo screening of the magnetic impurity can appear beyond some critical value of the Kondo coupling constant. Thus, the I-QPT from a local-moment state to a Kondo-screened phase was found in this model. Furthermore, the exponent r in the density of states was shown to play the role of an effective dimension in the problem. The $r = 1$ case was found to be its upper critical dimension, thus exhibiting logarithmic corrections to scaling while the case of $r = 0$ lies in its lower critical dimension.

However, there is an important difference between the pseudogap Kondo problem and ASL Kondo physics. The bulk physics in the pseudogap Kondo problem is governed by a non-interacting (Gaussian) fixed point, thus allowing well-defined electron-like quasiparticle excitations. On the other hand, the ASL physics is determined by an interacting fixed point (the conformal invariant fixed point of QED$_3$) [16], where well-defined spinon quasiparticle excitations corresponding to electrons do not exist. The absence of quasiparticle excitations prohibits us from applying the conventional picture of pseudogap Kondo physics to the ASL Kondo problem. In this respect the Kondo effect at such an interacting fixed point is an interesting problem.

The main difficulty is how to introduce the absence of well-defined spinon excitations in the ASL Kondo problem. Long-range gauge interactions would result in the anomalous critical exponent η_ψ in the single spinon propagator, destroying the quasiparticle pole in Green's function. Unfortunately, such critical physics can be found within the summation of infinite diagrams of gauge interactions, and this procedure prohibits us from analysing the ASL Kondo problem in a simple mean-field way such as the large-N_σ approximation with a spin index N_σ, well utilized in the pseudogap Kondo problem [13]. Considering the mathematical derivation in the large-N_σ context, the main problem is how to derive an effective impurity action from the ASL Kondo Lagrangian through integrating out bulk degrees of freedom, critical spinon and gauge fluctuations coupled to the magnetic impurity. More precisely speaking, a bulk-spinon propagator appears to govern the impurity dynamics in the effective impurity action, thus how to write its accurate form is an important problem since the presence of gauge interactions makes such a task nontrivial.

In the present paper we assume the expression of the spinon Green's function as an ansatz, introducing an anomalous critical exponent η_ψ. In the text we discuss the validity of this ansatz in great detail. This effective representation allows us to analyse the ASL Kondo problem in the large-N_σ context. Performing slave-boson saddle-point analysis for the effective impurity action, we find an I-QPT from a decoupled local-moment state to a Kondo-screened phase. We evaluate the impurity spin susceptibility and specific heat coefficient at zero temperature, and find that such responses follow power-law dependences due to the ASL anomalous exponent. The main finding of the present study is that Wilson's ratio for the magnetic impurity depends strongly on the ASL critical exponent in the zero temperature limit. We propose that Wilson's ratio for the magnetic impurity be a probe to reveal the criticality of the bulk system.

2. Review of the algebraic spin liquid and its Kondo problem

For completeness of this paper, it is necessary to review how the effective Lagrangian, the so-called QED$_3$ describing the ASL, is derived from a microscopic model such as the antiferromagnetic Heisenberg model, $H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$ with $J_{ij} > 0$. Inserting the fermion representation of spin $\vec{S}_i = \frac{1}{2} \sum_{\sigma\sigma'} f_{i\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} f_{i\sigma'}$ into the Heisenberg model, and performing the Hubbard–Stratonovich transformation for an exchange channel, we find an effective one-body Hamiltonian for fermionic spinons ($f_{i\sigma}$) coupled to a hopping parameter (χ_{ij}), $H_{\text{eff}} = -\sum_{ij\sigma} J_{ij} f_{i\sigma}^\dagger \chi_{ij} f_{j\sigma} + \sum_{ij} J_{ij} |\chi_{ij}|^2$. Notice that the hopping parameter χ_{ij} is a complex number defined on links ij. Thus, it can be decomposed into $\chi_{ij} = |\chi_{ij}| e^{i\theta_{ij}}$, where $|\chi_{ij}|$ and θ_{ij} are the amplitude and phase of the hopping parameter, respectively. Inserting this representation of χ_{ij} into the effective Hamiltonian, we obtain $H_{\text{eff}} = -\sum_{ij\sigma} J_{ij} |\chi_{ij}| f_{i\sigma}^\dagger e^{i\theta_{ij}} f_{j\sigma}$, where the constant contribution for the ground state energy is omitted. Then, we can see that this effective Hamiltonian has an internal U(1) gauge symmetry, $H'_{\text{eff}}[f'_{i\sigma}, \theta'_{ij}] = H_{\text{eff}}[f_{i\sigma}, \theta_{ij}]$ under the following U(1) phase transformation, $f'_{i\sigma} = e^{i\phi_i} f_{i\sigma}$ and $\theta'_{ij} = \theta_{ij} + \phi_i - \phi_j$. This implies that the phase θ_{ij} of the hopping parameter plays the same role as the U(1) gauge field a_{ij}.

One can perform a saddle-point analysis of the effective Hamiltonian to find its stable mean-field phases in various lattices such as square [12], triangular [6], Kagome [11, 17], etc. In the present paper we consider the square lattice for simplicity, where the antiferromagnetic long-range order can be suppressed via next-nearest-neighbour or ring-exchange interactions causing frustration. It is not so difficult to extend the mean-field analysis on the square lattice into that on the Kagome lattice, proposed to show the SL physics of ZnCu$_3$(OH)$_6$Cl$_2$ [11].

It has been shown that one possible stable mean-field phase is a π-flux state, where a spinon gains the phase of π when it turns around one plaquette. The amplitude of the hopping parameter is frozen to be $|\chi_{ij}| = \sum_\sigma |\langle f_{j\sigma}^\dagger f_{i\sigma} \rangle| \equiv \chi_0$ in the low-energy limit. Then, one finds the low-energy effective Lagrangian in terms of massless Dirac fermions

interacting via compact U(1) gauge fields [18]

$$Z = \int D\psi_{n\sigma} Da_\mu e^{-\int d^3 x \mathcal{L}},$$

$$\mathcal{L} = \sum_{\sigma=\uparrow,\downarrow} \sum_{n=1}^{2} \bar{\psi}_{n\sigma} \gamma_\mu (\partial_\mu - i a_\mu) \psi_{n\sigma} + \frac{1}{2e^2} |\partial \times a|^2. \quad (1)$$

Here, $\psi_{n\sigma}$ is the two-component massless Dirac spinon, where $n = 1, 2$ represent the nodal points of $(\pi/2, \pi/2)$ and $(\pi/2, -\pi/2)$, and $\sigma = \uparrow, \downarrow$, SU(2) spin. They are expressed as $\psi_{1\sigma} = \binom{f_{1e\sigma}}{f_{1o\sigma}}$ and $\psi_{2\sigma} = \binom{f_{2o\sigma}}{f_{2e\sigma}}$, respectively. In the spinon field $f_{nl\sigma}$ $n = 1, 2$ represent the nodal points, $l = e, o$, even and odd sites, and $\sigma = \uparrow, \downarrow$, its spin, respectively. The Dirac matrices γ_μ are given by the Pauli matrices $\gamma_\mu = (\sigma_3, \sigma_2, \sigma_1)$, satisfying the Clifford algebra $[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu}$. a_μ is the U(1) gauge field whose kinetic energy results from particle–hole excitations of high-energy spinons. e is an effective internal charge, not a real electric charge.

It has been argued that QED$_3$ has an infrared stable fixed point showing the conformal symmetry in the large-N_σ limit ($\sigma = 1, \ldots, N_\sigma$) [16]. This conformal invariant fixed point is identified with the ASL, displaying algebraically decaying correlation functions with anomalous critical exponents. To confirm the ASL as a genuine stable phase a cautious person may query the stability of such an interacting fixed point against perturbations. Four-fermion interaction terms are irrelevant at this fixed point owing to their high scaling dimensions. In addition, chiral symmetry breaking due to noncompact gauge fluctuations has been shown not to occur in the Schwinger–Dyson-equation analysis when the flavour number of massless Dirac fermions is sufficiently large [19]. Furthermore, it has been argued that confinement as an instanton effect arising from compact gauge fluctuations does not seem to appear in the large-N_σ limit because the scaling dimension of the monopole insertion operator is proportional to the flavour number N_σ, thus expected to be irrelevant in the large-N_σ ASL [16].

Criticality of the ASL is characterized by critical exponents of correlation functions. The single particle propagator $G_{\mathrm{ASL}}(k) = \langle \psi_{n\sigma}(k) \bar{\psi}_{n\sigma}(k) \rangle$ can be expressed as

$$G_{\mathrm{ASL}}(k) \approx -i \frac{\gamma_\mu k_\mu}{k^{2-\eta_\psi}}, \quad (2)$$

where η_ψ is an anomalous critical exponent. One can find such an anomalous dimension in the large-N_σ analysis[1]. However,

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[1] The renormalized spinon propagator is expressed as $G_{\mathrm{ASL}}^{-1}(k) = G_0^{-1}(k) + \Sigma(k)$, where $G_0^{-1}(k) = i\gamma_\mu k_\mu$ is the inverse of the bare spinon propagator, and $\Sigma(k)$ the spinon self-energy resulting from long-range gauge interactions. In the $1/N_\sigma$ expansion the self-energy is represented as $\Sigma(k) = \int \frac{d^3 q}{(2\pi)^3} \gamma_\mu G_0(k+q) \gamma_\nu D_{\mu\nu}(q)$, where $D_{\mu\nu}(q)$ is the renormalized propagator of the U(1) gauge field due to polarization of massless Dirac fermions. The gauge propagator is obtained to be $D_{\mu\nu}(q) \approx \Pi_{\mu\nu}^{-1}(q) = \frac{8}{N_\sigma q}(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})$ in the Lorentz gauge, where $\Pi_{\mu\nu}(q) = N_\sigma \int \frac{d^3 k}{(2\pi)^3} Tr[G_0(k)\gamma_\mu G_0(k+q)\gamma_\nu]$ is the polarization function of Dirac fermions. Inserting this gauge propagator into the expression of the self-energy, one can find the spinon self-energy of logarithmic momentum dependence $\Sigma(k) = i\eta_\psi \gamma_\mu k_\mu \ln(\frac{\Lambda}{k})$, where η_ψ is the anomalous exponent and Λ, the momentum cutoff. The absolute value of the exponent η_ψ is proportional to the inverse of the flavour number,

it is difficult to give the critical exponent obtained in this way a definite physical meaning because it is not gauge invariant. In this respect the critical exponent η_ψ should be evaluated in a gauge invariant way. The following gauge invariant Green's function can be considered, $G_{\mathrm{ASL}}(x) = \langle T_\tau[\psi_{n\sigma}(x)e^{i\int_0^x d\zeta_\mu a_\mu(\zeta)}\bar{\psi}_{n\sigma}(0)]\rangle$. Unfortunately, it is not easy to calculate the critical exponent with such a gauge invariant expression. Its precise value is far from consensus and still under current debate. The crucial point is the sign of the exponent η_ψ while its absolute value is given by $|\eta_\psi| \sim N_\sigma^{-1}$ in the $1/N_\sigma$ approximation (see footnote 1). Most evaluations [20–23] suggest its negative sign, $\eta_\psi < 0$. However, as argued in [24], its negative sign seems to be unphysical in the sense that the spinon propagator becomes more 'coherent' at longer distances than the propagator of the free Dirac theory. This result is in contrast with the usual role of interactions, making elementary excitations less coherent. This is indeed true in such critical field theories with local repulsive interactions, for example, the N-vector model, where positive critical exponents are well known [25]. If the critical exponent is positive, long-range gauge interactions destabilize the quasiparticle pole. The quasiparticle weight $Z(p) \sim p^{\eta_\psi}$ with momentum p vanishes in the long-wavelength and low-energy limits. In the present paper we do not determine its sign. Instead, we regard the exponent η_ψ as a phenomenological parameter. Thus, we consider both cases of $\eta_\psi < 0$ and $\eta_\psi > 0$. Furthermore, we assume that the renormalized spinon propagator (equation (2)) is obtained in a gauge invariant way [20–24], and the critical exponent η_ψ is also gauge invariant.

Another important character of the ASL is that the conformally invariant fixed point has an enlarged global symmetry beyond the original lattice model, here the Heisenberg Hamiltonian. Such an emergent symmetry corresponds to Sp(4) in the case of SU(2) gauge interactions [26] and SU(4) in the case of U(1) ones [27]. This enlarged symmetry gives rise to an important effect on correlation functions, that is, resulting in the same behaviours between different correlation functions when the operators in the correlators are related with symmetry transformations. For example, staggered spin correlations have the same functional dependency (power-law decay) as the valance bond fluctuations since they are symmetry-equivalent. An interesting point is that such correlations are most susceptible in the ASL [27]. This implies that the ASL resides near the antiferromagnetic and valence bond solid phases. Actually, Tanaka and Hu have derived an effective Wess–Zumino–Witten (WZW) Lagrangian from the ASL, describing competition between antiferromagnetic spin correlations and valence bond fluctuations [28].

To study the role of magnetic impurities in the ASL bulk, we consider the Kondo coupling term, $H_{\mathrm{K}} = \frac{J_{\mathrm{K}}}{2} \sum_q \vec{S}_q \cdot \vec{s}$, where $\vec{S}_q$ is a spin-fluctuation operator of bulk spinons with

i.e. $|\eta_\psi| \sim N_\sigma^{-1}$. Such logarithmic momentum dependence should be considered as the lowest order in the algebraic function. As a result, one can obtain the algebraically decaying spinon propagator (equation (2)) from the following nonperturbative consideration $G_{\mathrm{ASL}}^{-1}(k) = i\gamma_\mu k_\mu[1 + \eta_\psi \ln(\frac{\Lambda}{k})] \approx i\gamma_\mu k_\mu(\frac{\Lambda}{k})^{\eta_\psi}$.

momentum q and $\vec{s}$ represents an impurity spin. The bulk-spin operator has two contributions in the continuum,

$$
\vec{S}(q) \approx \vec{S}_{\rm u}(q) + \vec{S}_{\rm s}(q) = \sum_k \sum_{n\sigma\sigma'} \bar{\psi}_{n\sigma}(k-q)\gamma_0 \frac{\vec{\tau}_{\sigma\sigma'}}{2}\psi_{n\sigma'}(k)
$$
$$
+ \sum_k \sum_{n\sigma\sigma'} \bar{\psi}_{n\sigma}(k-q)\frac{\vec{\tau}_{\sigma\sigma'}}{2}\psi_{n\sigma'}(k), \tag{3}
$$

where $\vec{S}_{\rm u}(q)$ represents the uniform component and $\vec{S}_{\rm s}(q)$ denotes the staggered one [18]. Then, the ASL Kondo problem is described by the following action

$$
S = \int {\rm d}\tau \left\{ \int {\rm d}^2 r \left(\sum_{\sigma=\uparrow,\downarrow}\sum_{n=1}^{2} \bar{\psi}_{n\sigma}\gamma_\mu(\partial_\mu - {\rm i}a_\mu)\psi_{n\sigma} \right. \right.
$$
$$
\left. \left. + \frac{1}{2e^2}|\partial\times a|^2 \right) + \frac{J_{\rm K}}{2}\sum_q (\vec{S}_{\rm u}(q) + \vec{S}_{\rm s}(q))\cdot\vec{s} \right\}. \tag{4}
$$

The next task is to obtain an effective impurity action, integrating out bulk degrees of freedom, spinon and gauge excitations coupled to the magnetic impurity. One can write down its schematic expression in the following way

$$
S_{\rm imp} \approx -\frac{J_{\rm K}^2}{4}\int {\rm d}\tau\, {\rm d}\tau'\, s^a(\tau)\left(\sum_q \langle S_{\rm u}^a(q,\tau)S_{\rm u}^b(-q,\tau')\rangle \right.
$$
$$
\left. + \sum_q \langle S_{\rm s}^a(q,\tau)S_{\rm s}^b(-q,\tau')\rangle \right) s^b(\tau') + \cdots, \tag{5}
$$

where $\langle S_{{\rm s}({\rm u})}^a(q,\tau)S_{{\rm s}({\rm u})}^b(-q,\tau')\rangle$ is the renormalized correlation function of staggered (uniform) spin fluctuations and $\cdots$ are higher moment contributions. As clearly shown in this expression, dynamics of impurity spin fluctuations are governed by spin correlations of the bulk at the impurity site. An important point is that only staggered spin correlations exhibit an anomalous scaling behaviour with a nontrivial critical exponent [29, 30]. Uniform spin correlations have no anomalous scaling dimension since they correspond to conserved currents [18, 29]. Correlations of conserved currents do not have any anomalous scaling dimensions. This means that the contribution of uniform spin fluctuations is basically the same as the Kondo effect of the pseudogap Kondo model while that of staggered spin excitations will give rise to new effects on the pseudogap Kondo physics. Furthermore, staggered spin fluctuations are most singular in the large-N_σ ASL [27], thus expected to contribute to the Kondo effect dominantly. In this respect we take into account staggered spin fluctuations only, which is an important assumption in the present paper.

3. Kondo physics in the algebraic spin liquid: large-N_σ analysis

Our objective is to construct a mean-field theory for the present Kondo problem. Using the slave-boson representation, the impurity spin is expressed as $\vec{s} = \frac{1}{2}\sum_{\sigma\sigma'}\chi_\sigma^\dagger \vec{\tau}_{\sigma\sigma'}\chi_{\sigma'}$, and such fermions satisfy the constraint $\sum_{\sigma=1}^{N_\sigma}\chi_\sigma^\dagger\chi_\sigma = Q_\chi$ with $Q_\chi = 2s$, where s is spin. Inserting this expression into equation (4)

with equation (3), the Kondo coupling term becomes

$$
H_{\rm K} = -\frac{J_{\rm K}}{2N_\sigma}\sum_{qk}\sum_{n=1}^{N_n}\sum_{\sigma\sigma'=1}^{N_\sigma}\bar{\psi}_{n\sigma}(k-q)\chi_\sigma\chi_{\sigma'}^\dagger\gamma_0\psi_{n\sigma'}(q)
$$
$$
-\frac{J_{\rm K}}{2N_\sigma}\sum_{qk}\sum_{n=1}^{N_n}\sum_{\sigma\sigma'=1}^{N_\sigma}\bar{\psi}_{n\sigma}(k-q)\chi_\sigma\chi_{\sigma'}^\dagger\psi_{n\sigma'}(q) \tag{6}
$$

in the large-N_σ treatment, where the first and second terms are associated with uniform and staggered spin-fluctuation contributions, respectively. Since staggered spin fluctuations will give the main contributions to the ASL Kondo effect, effects of uniform spin fluctuations are neglected in the following.

Performing the Hubbard–Stratonovich transformation for the Kondo-exchange channel, we find an effective ASL Kondo action as our starting point

$$
S_{\rm eff} = \int {\rm d}^3 x \left[\left(\sum_{\sigma=1}^{N_\sigma}\sum_{n=1}^{N_n}\bar{\psi}_{n\sigma}\gamma_\mu(\partial_\mu - {\rm i}a_\mu)\psi_{n\sigma} + \frac{1}{2e^2}|\partial\times a|^2 \right) \right.
$$
$$
- \sum_{\sigma=1}^{N_\sigma}\sum_{n=1}^{N_n}(b_n^{s\dagger}\chi_\sigma^\dagger\psi_{n\sigma}(0) + \bar{\psi}_{n\sigma}(0)\gamma_0\chi_\sigma b_n^s) \right]
$$
$$
+ \int {\rm d}\tau \left[\sum_{\sigma=1}^{N_\sigma}\chi_\sigma^\dagger(\partial_\tau - h_\sigma)\chi_\sigma + {\rm i}\lambda\left(\sum_{\sigma=1}^{N_\sigma}\chi_\sigma^\dagger\chi_\sigma - Q_\chi \right) \right.
$$
$$
\left. + \frac{N_\sigma}{2J_{\rm K}}\sum_{n=1}^{N_n}b_n^{s\dagger}\gamma_0 b_n^s \right]. \tag{7}
$$

The first part represents the ASL bulk. The second part arises from the Hubbard–Stratonovich decoupling of the Kondo interaction term, where b_n^s is a two-component hybridization order parameter associated with staggered bulk-spin fluctuations. Such a hybridization order parameter is determined self-consistently in the saddle-point analysis

$$
\frac{N_\sigma}{2J_{\rm K}}\gamma_0 b_n^s = \left\langle \int \frac{{\rm d}^2 k}{(2\pi)^2}\sum_{\sigma=1}^{N_\sigma}\chi_\sigma^\dagger\psi_{n\sigma}(k) \right\rangle. \tag{8}
$$

The third part describes impurity-spinon dynamics, where $h_\sigma = \sigma H$ is an external magnetic field and λ is a Lagrange multiplier field to impose the pseudo-fermion constraint.

When the bulk system is in the non-interacting fixed point corresponding to the absence of gauge interactions, the effective Kondo model becomes the multi-channel pseudogap Kondo model, where the channels come from Dirac nodes $n = 1, \ldots, N_n$. This model was argued to show an I-QPT from a decoupled local-moment state to an over-screened phase in the large-N_σ approximation although this analysis does not capture the over-screened Kondo physics very well [13]. On the other hand, the present bulk system lies at the interacting fixed point characterized by the anomalous critical exponent η_ψ, where quasiparticle excitations do not exist. In this case it is not clear whether the conventional Kondo screening picture is applicable.

Integrating out bulk-spinon and gauge excitations, we obtain an effective impurity action in energy–momentum space

$$S_{\text{eff}}^{\text{imp}} = \int \frac{dk_0}{2\pi} \left[\sum_{\sigma=1}^{N_\sigma} \chi_\sigma^\dagger (ik_0 - h_\sigma + \epsilon_\chi) \chi_\sigma \right.$$

$$- \sum_{\sigma=1}^{N_\sigma} \sum_{n=1}^{N_n} b_n^{s\dagger} \chi_\sigma^\dagger \left(\int \frac{d^2k}{(2\pi)^2} \langle \psi_{n\sigma}(k) \bar{\psi}_{n\sigma}(k) \rangle \right) \gamma_0 \chi_\sigma b_n^s$$

$$\left. + \frac{N_\sigma}{2J_K} \sum_{n=1}^{N_n} b_n^{s\dagger} \gamma_0 b_n^s - \epsilon_\chi Q_\chi \right], \tag{9}$$

where $i\lambda$ is replaced with ϵ_χ to clarify its physical meaning. The main question in this impurity action is how to evaluate the spinon Green's function. As discussed intensively in the previous section, the single particle propagator has an anomalous scaling exponent, given by $\langle \psi_{n\sigma}(k) \bar{\psi}_{n\sigma}(k) \rangle = -i\gamma_\mu k_\mu / |k|^{2-\eta_\psi}$. This expression seems to be consistent with equation (5) if 'sub-leading' uniform spin-correlation contributions are not taken into account. This is because the critical exponent of the staggered spin–spin correlation function is found to be twice the exponent of the single particle propagator, i.e. $2\eta_\psi$ in the case of $\eta_\psi < 0$ [29, 30]. Such correspondence occurs when both critical exponents are calculated in a gauge invariant way. This correspondence was also pointed out in [23].

Such spinon excitations with an anomalous scaling exponent result in anomalous energy-dependent (nonlocal in time) interactions for impurity fermions, as reflected in the kernel of $\int \frac{d^2k}{(2\pi)^2} \frac{i\gamma_\mu k_\mu}{|k|^{2-\eta_\psi}} \equiv i\gamma_\mu F_\mu(k_0)$. The vector function $F_\mu(k_0)$ is obtained to be

$$F_\mu(k_0) = \int \frac{d^2k}{(2\pi)^2} \frac{k_\mu}{(k_0^2 + k^2)^{1-\eta_\psi/2}}$$

$$= k_0 \frac{(k_0^2 + \Lambda^2)^{\eta_\psi/2} - |k_0|^{\eta_\psi}}{2\pi\eta_\psi} \delta_{\mu 0}, \tag{10}$$

thus $i\gamma_\mu F_\mu(k_0) = i\gamma_0 k_0 F(k_0)$ with $F(k_0) = [(k_0^2 + \Lambda^2)^{\eta_\psi/2} - |k_0|^{\eta_\psi}]/2\pi\eta_\psi$, where Λ is a momentum cutoff.

Inserting equation (10) into equation (9) and integrating over impurity fermions in equation (9), we find the following expression for the impurity free energy

$$F_{\text{imp}} = -\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \sum_{\sigma=1}^{N_\sigma} \ln \left[(ik_0 - h_\sigma + \epsilon_\chi) \right.$$

$$\left. + ik_0 F(k_0) \sum_{n=1}^{N_n} b_n^{s\dagger} b_n^s \right] + \frac{N_\sigma}{2J_K} \sum_{n=1}^{N_n} b_n^{s\dagger} \gamma_0 b_n^s - \epsilon_\chi Q_\chi. \tag{11}$$

Expressing the hybridization order parameter as a two-component spinor $b_n^{s\dagger} = (b_{n+}^{s\dagger} b_{n-}^{s\dagger})$, one can find $b_{n-}^s = 0$ in the saddle-point analysis. Representing the above impurity free energy with $b_{n+}^s \equiv 2b$, we obtain

$$F_{\text{imp}} = -\frac{N_\sigma}{2} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left[\ln \left(ik_0 - H + \epsilon_\chi \right. \right.$$

$$\left. + ik_0 \frac{2N_n}{\pi\eta_\psi} |b|^2 \{(k_0^2 + \Lambda^2)^{\eta_\psi/2} - |k_0|^{\eta_\psi}\} \right)$$

$$+ \ln \left(ik_0 + H + \epsilon_\chi + ik_0 \frac{2N_n}{\pi\eta_\psi} |b|^2 \{(k_0^2 + \Lambda^2)^{\eta_\psi/2} \right.$$

$$\left. \left. - |k_0|^{\eta_\psi}\} \right) \right] + \frac{2N_\sigma N_n}{J_K} |b|^2 - \epsilon_\chi Q_\chi. \tag{12}$$

Minimizing the impurity free energy with respect to b and ϵ_χ, we find the saddle-point equations giving the self-consistency

$$b \left(\frac{1}{J_K} - \int_0^\infty \frac{dk_0}{2\pi} \left\{ k_0^2 \frac{2}{\pi\eta_\psi} \{(k_0^2 + \Lambda^2)^{\eta_\psi/2} - k_0^{\eta_\psi}\} \right. \right.$$

$$\times \left[1 + \frac{2N_n}{\pi\eta_\psi} |b|^2 \{(k_0^2 + \Lambda^2)^{\eta_\psi/2} - k_0^{\eta_\psi}\} \right] \right\}$$

$$\times \left. \left\{ k_0^2 \left[1 + \frac{2N_n}{\pi\eta_\psi} |b|^2 \{(k_0^2 + \Lambda^2)^{\eta_\psi/2} - k_0^{\eta_\psi}\} \right]^2 + \epsilon_\chi^2 \right\}^{-1} \right)$$

$$= 0,$$

$$\frac{Q_\chi}{N_\sigma} =$$

$$- \int_0^\infty \frac{dk_0}{2\pi} \frac{2\epsilon_\chi}{k_0^2 \left[1 + \frac{2N_n}{\pi\eta_\psi} |b|^2 \{(k_0^2 + \Lambda^2)^{\eta_\psi/2} - k_0^{\eta_\psi}\} \right]^2 + \epsilon_\chi^2}. \tag{13}$$

Since the impurity free energy is momentum-cutoff-dependent, it is necessary to make it cutoff-independent, taking appropriate scaling transformations for all variables. Considering the scaling dimension of $\psi_{n\sigma}$ given by $\dim[\psi_{n\sigma}] = 1 + \eta_\psi/2$, one can find $\dim[b] = -\eta_\psi/2$ and $\dim[J_K] = -1 - \eta_\psi$, where $\dim[\hat{\mathcal{O}}]$ represents the scaling dimension of an operator $\hat{\mathcal{O}}$. Then, the scale-free impurity free energy is obtained to be

$$f_{\text{imp}} \equiv \frac{F_{\text{imp}}}{\Lambda} = -\frac{N_\sigma}{4\pi} \int_{-\infty}^\infty dx \left[\ln \left(ix - h + \epsilon_r \right. \right.$$

$$\left. + ix \frac{2N_n}{\pi\eta_\psi} |b_r|^2 \{(x^2 + 1)^{\eta_\psi/2} - |x|^{\eta_\psi}\} \right)$$

$$+ \ln \left(ix + h + \epsilon_r + ix \frac{2N_n}{\pi\eta_\psi} |b_r|^2 \{(x^2 + 1)^{\eta_\psi/2} \right.$$

$$\left. \left. - |x|^{\eta_\psi}\} \right) \right] + \frac{2N_\sigma N_n}{J_r} |b_r|^2 - \epsilon_r Q_\chi, \tag{14}$$

where such rescaled variables are given by

$$b_r = \frac{b}{\Lambda^{-\eta_\psi/2}}, \qquad J_r = \frac{J_K}{\Lambda^{-(1+\eta_\psi)}},$$

$$\epsilon_r = \frac{\epsilon_\chi}{\Lambda}, \qquad x = \frac{k_0}{\Lambda}, \qquad h = \frac{H}{\Lambda}.$$

Notice that these scaled variables are dimensionless. Accordingly, the self-consistent saddle-point equations read

$$\frac{1}{J_r} = \frac{1}{\pi^2} \int_0^\infty dx \left\{ \frac{x^2}{\eta_\psi} \{(x^2 + 1)^{\eta_\psi/2} - x^{\eta_\psi}\} \right.$$

$$\times \left[1 + \frac{2N_n}{\pi\eta_\psi} |b_r|^2 \{(x^2 + 1)^{\eta_\psi/2} - x^{\eta_\psi}\} \right] \right\}$$

$$\times \left\{ x^2 \left[1 + \frac{2N_n}{\pi\eta_\psi} |b_r|^2 \{(x^2 + 1)^{\eta_\psi/2} - x^{\eta_\psi}\} \right]^2 + \epsilon_r^2 \right\}^{-1},$$

$$\frac{Q_\chi}{N_\sigma} = -\frac{1}{\pi}$$

$$\times \int_0^\infty dx \frac{\epsilon_r}{x^2 \left[1 + \frac{2N_n}{\pi\eta_\psi} |b_r|^2 \{(x^2 + 1)^{\eta_\psi/2} - x^{\eta_\psi}\} \right]^2 + \epsilon_r^2}. \tag{15}$$

The QCP of the I-QPT can be found with $b_r \to 0$ and $\epsilon_r \to 0$ in the particle–hole symmetric case, $Q_\chi/N_\sigma = 1/2$.
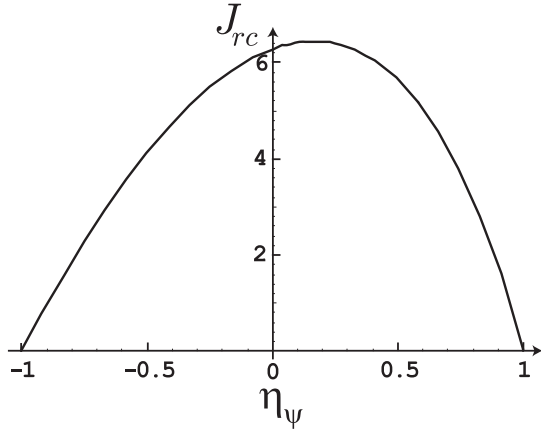
Figure 1. Renormalized critical Kondo coupling constant as a function of the ASL exponent.

Then, the critical renormalized Kondo coupling constant is obtained from equation (15),

$$
\frac{1}{J_{rc}} = \frac{1}{\pi^2 \eta_\psi} \int_0^\infty dx \left\{ (x^2+1)^{\eta_\psi/2} - x^{\eta_\psi} \right\}
$$
$$
= \frac{1}{2\pi^{3/2}\eta_\psi} \frac{\Gamma\left(-\frac{1+\eta_\psi}{2}\right)}{\Gamma\left(-\frac{\eta_\psi}{2}\right)}, \tag{16}
$$

as far as the ASL exponent lies in $-1 < \eta_\psi < 1$. In the large N_σ limit the ASL exponent may also satisfy this condition as discussed in section 2. In addition, this critical value is continuously defined in the limit of $\eta_\psi \to \pm 0$, where the impurity critical point[2] is given by

$$
\frac{1}{J_{rc}} = \frac{1}{2\pi^2} \int_0^\infty dx \ln\left(1 + \frac{1}{x^2}\right) = \frac{1}{2\pi}, \tag{17}
$$

consistent with the previous study [13].

It is interesting to notice that the I-QPT occurs as long as the ASL exponent $|\eta_\psi| < 1$. Remember that in the regime of $0 < \eta_\psi < 1$ critical spinon excitations are less coherent than those in the pseudogap Kondo model ($\eta_\psi = 0$) while in the regime of $-1 < \eta_\psi < 0$ such spinon excitations become more coherent than quasiparticle excitations in the Fermi liquid with pseudogap. To screen the magnetic impurity, stronger Kondo couplings would be required when quasiparticle excitations are less coherent. Actually, we find such an asymmetric behavior for the ASL exponent in figure 1, obtained from equation (16).

It might seem mysterious that the critical Kondo coupling vanishes as $\eta_\psi \to \pm 1$. As the ASL exponent approaches 1, critical spinon excitations are not only less coherent but also localized. Considering the spinon propagator equation (2),

---

[2] The saddle-point equations of the pseudogap Kondo model can be obtained from equation (14) in the limit of $\eta_\psi \to 0$

$$
\frac{1}{J_r} = \frac{1}{2\pi^2} \int_0^\infty dx \frac{x^2 \ln\left(1+\frac{1}{x^2}\right)\left[1 + \frac{N_n}{\pi}|b_r|^2 \ln\left(1+\frac{1}{x^2}\right)\right]}{x^2\left[1 + \frac{N_n}{\pi}|b_r|^2 \ln\left(1+\frac{1}{x^2}\right)\right]^2 + \epsilon_r^2},
$$
$$
\frac{Q_\chi}{N_\sigma} = -\frac{1}{\pi} \int_0^\infty dx \frac{\epsilon_r}{x^2\left[1 + \frac{N_n}{\pi}|b_r|^2 \ln\left(1+\frac{1}{x^2}\right)\right]^2 + \epsilon_r^2},
$$

giving rise to equation (16) for the impurity QCP.

$\eta_\psi = 1$ makes it energy–momentum-independent. Such localized spinons are expected to form a Kondo singlet with an impurity spin immediately. When the ASL exponent goes to -1, it is important that the bare scaling dimension of the Kondo coupling ($\dim[J_K] = -1 - \eta_\psi$) vanishes, implying that Kondo interactions are marginal perturbations similar to the conventional Kondo effect in the Fermi liquid. In this respect the critical Kondo coupling would go to zero as $\eta_\psi \to -1$.

Solving equation (15) numerically, one can find the hybridization amplitude $|b_r|^2$ as a function of the Kondo coupling J_r. We show the I-QPT in figure 2, where both b_r and ϵ_r vanish as $J_r \to J_{rc}$. It is important to notice that the x-axis is $J_r - J_{rc}$ instead of J_r. This means that the impurity QCP matches the origin of the x-axis. The absolute value of the impurity chemical potential increases rapidly as the ASL exponent increases from $\eta_\psi = -0.2$ to 0.2 (figure 2(a)). Accordingly, the increasing ratio of the hybridization order parameter is largest for $\eta_\psi = 0.2$ and smallest for $\eta_\psi = -0.2$. This may be associated with localization tendency emerging from a positive exponent. A further analysis finds a scaling behavior of the hybridization amplitude not only near the impurity QCP, but also further away from the QCP, i.e. in the Kondo-screened phase. Such a scaling behavior even in the Kondo phase seems to arise from the criticality of the bulk system. From the log–log plot of figure 2(b), we find the scaling relation

$$
|b_r|^2 \sim (J_r - J_{rc})^{f(\eta_\psi)} \tag{18}
$$

with $f(\eta_\psi) \approx 3 + 2\eta_\psi$, confirming that the slope of the positive ASL exponent is larger than that of the negative one.

The I-QPT can be also found in the impurity-spin susceptibility,

$$
\chi_{imp} = -\frac{\partial^2 f_{imp}(h)}{\partial h^2} = -\frac{N_\sigma}{\pi} \int_0^\infty dx
$$
$$
\times \frac{\epsilon_r^2 - x^2\left[1 + \frac{2N_n}{\pi\eta_\psi}|b_r|^2\left\{(x^2+1)^{\eta_\psi/2} - x^{\eta_\psi}\right\}\right]^2}{\left(x^2\left[1 + \frac{2N_n}{\pi\eta_\psi}|b_r|^2\left\{(x^2+1)^{\eta_\psi/2} - x^{\eta_\psi}\right\}\right]^2 + \epsilon_r^2\right)^2}. \tag{19}
$$

In the decoupled phase ($J_r < J_{rc}$) the impurity susceptibility diverges in the zero temperature limit (following the Curie law) while it vanishes in the screened phase. Since for $\eta_\psi = -0.2$ the hybridization amplitude is smallest, the impurity-spin susceptibility becomes largest. Approaching the impurity QCP ($J_r \to J_{rc}$), it shows a power-law divergence with an anomalous critical exponent of the ASL bulk. As shown in figure 3, such curves are well fitted with

$$
\chi_{imp} \sim (J_r - J_{rc})^{-g(\eta_\psi)}, \tag{20}
$$

where the scaling function is $g(\eta_\psi) \approx 2 - \eta_\psi$. It is valuable to consider how the behavior of the impurity susceptibility differs from that of the pseudogap Kondo model [3] which corresponds to the case of $\eta_\psi = 0$.

---

[3] In the pseudogap Kondo model the impurity susceptibility is given by

$$
\chi_{imp} = -\frac{N_\sigma}{\pi} \int_0^\infty dx \frac{\epsilon_r^2 - x^2\left[1 + \frac{N_n}{\pi}|b_r|^2 \ln\left(1+\frac{1}{x^2}\right)\right]^2}{\left(x^2\left[1 + \frac{N_n}{\pi}|b_r|^2 \ln\left(1+\frac{1}{x^2}\right)\right]^2 + \epsilon_r^2\right)^2}.
$$

Remember that the impurity susceptibility based on the large-N_σ analysis exhibits a logarithmic correction due to the upper critical dimensionality [13].
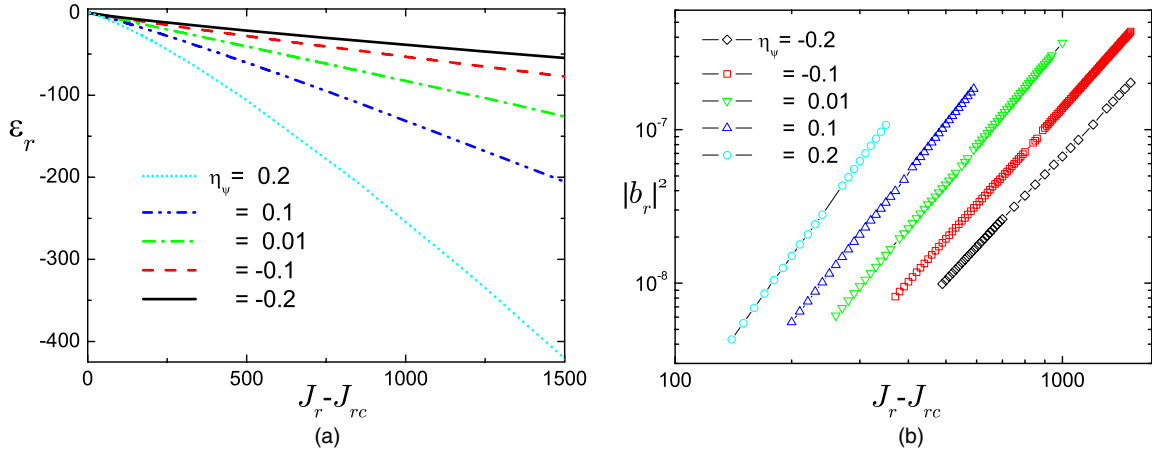
Figure 2. Impurity chemical potential, ϵ_r, and hybridization amplitude, b_r, (log–log plot) as a function of the renormalized Kondo coupling, J_r, for various ASL exponents. Here ϵ_r, b_r, and J_r are dimensionless rescaled variables.
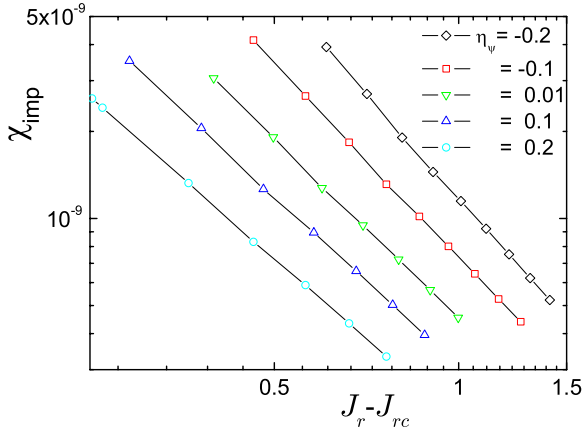

Figure 3. Impurity spin susceptibility (log–log plot) as a function of the renormalized Kondo coupling for various ASL exponents.
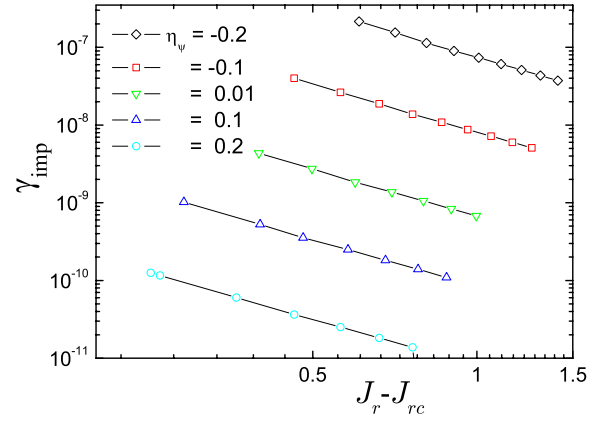
Figure 4. Specific heat coefficient (log–log plot) as a function of the renormalized Kondo coupling for various ASL exponents.

Next, we evaluate the impurity specific heat. The zero temperature formulation (equation (14)) for the impurity free energy can be transformed to the finite temperature version through the Wick rotation. Following [13, 31], we find the impurity free energy at finite temperatures,

$$f_{\rm imp} = N_\sigma \int_{-\infty}^{\infty} \frac{{\rm d}\xi}{\pi} \frac{1}{{\rm e}^{\beta_r \xi} + 1} \Theta(\xi) + \frac{2 N_\sigma N_n}{J_r} |b_r|^2 - \epsilon_r Q_\chi$$

with a rescaled temperature $\beta_r^{-1} = T_r = T/\Lambda$, where the 'angle' function $\Theta(\xi)$ is given by

$$\Theta(\xi) = \tan^{-1}\left(\left\{\frac{2 N_n}{\pi \eta_\psi} |b_r|^2 \sin\left(\frac{\pi \eta_\psi}{2}\right) |\xi|^{1+\eta_\psi}\right\}\right.$$
$$\times \left\{\xi\left[1 + \frac{2 N_n}{\pi \eta_\psi} |b_r|^2 \left\{(-\xi^2 + 1)^{\eta_\psi/2}\right.\right.\right.$$
$$\left.\left.\left.- \cos\left(\frac{\pi \eta_\psi}{2}\right) |\xi|^{\eta_\psi}\right\}\right] + \epsilon_r\right\}^{-1}\right)$$
$$+ \frac{\pi}{2}(1 - {\rm sign}(\xi)) \qquad \text{for } |\xi| < 1,$$
$$= \tan^{-1}\left(\left\{\frac{2 N_n}{\pi \eta_\psi} |b_r|^2 \sin\left(\frac{\pi \eta_\psi}{2}\right) \xi\left\{(\xi^2 - 1)^{\eta_\psi/2} - |\xi|^{\eta_\psi}\right\}\right\}\right.$$

$$\times \left\{\xi\left[1 + \frac{2 N_n}{\pi \eta_\psi} |b_r|^2 \cos\left(\frac{\pi \eta_\psi}{2}\right)\left\{(\xi^2 - 1)^{\eta_\psi/2} - |\xi|^{\eta_\psi}\right\}\right]\right.$$
$$\left.+ \epsilon_r\right\}^{-1}\right) + \frac{\pi}{2}(1 - {\rm sign}(\xi)) \qquad \text{for } |\xi| \geqslant 1. \qquad (21)$$

Here, the denominator and numerator in the angle function correspond to the real and imaginary parts of the kernel for the impurity free energy (equation (14)), respectively.

We find the impurity entropy

$$S_{\rm imp} = -\frac{\partial f_{\rm imp}}{\partial T_r}\bigg|_{\epsilon_r, b_r} = N_\sigma \int_{-\infty}^{\infty} \frac{{\rm d}\xi}{\pi} \frac{\xi}{T_r} \frac{\partial}{\partial \xi}\left(\frac{1}{{\rm e}^{\beta_r \xi} + 1}\right) \Theta(\xi)$$
$$(22)$$

and specific heat coefficient

$$\gamma_{\rm imp} = \frac{C_{\rm imp}}{T_r} = \frac{\partial S_{\rm imp}}{\partial T_r}$$
$$= \frac{N_\sigma}{T_r^2} \int_{-\infty}^{\infty} \frac{{\rm d}\xi}{\pi} \xi^2 \frac{\partial}{\partial \xi}\left(\frac{1}{{\rm e}^{\beta_r \xi} + 1}\right) \frac{\partial \Theta(\xi)}{\partial \xi}. \qquad (23)$$

Taking the zero temperature limit, we find the self-consistent results in figure 4, using the solutions of equation (15). The latter terms in equation (21) ensure that the impurity entropy
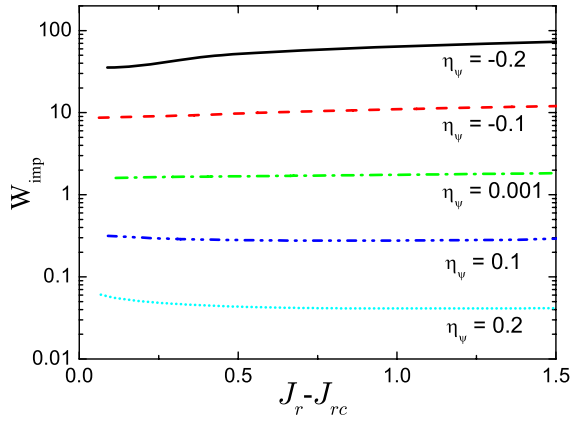
Figure 5. Wilson's ratio (y axis-log plot) as a function of the renormalized Kondo coupling for various ASL exponents.

is $S_{imp}/N_\sigma = \ln 2$ in the decoupled phase, consistent with our expectation. In the Kondo phase the impurity entropy vanishes, but we note that more elaborate calculations result in small nonzero entropy contributions in the Kondo phase [32, 33]. The γ_{imp} coefficient shows a behaviour similar to the impurity susceptibility χ_{imp}, diverging as $J_r \rightarrow J_{rc}$. It exhibits the scaling behaviour

$$\gamma_{imp} \sim (J_r - J_{rc})^{-h(\eta_\psi)} \tag{24}$$

with $h(\eta_\psi) \approx 2 - 0.2\eta_\psi$ in our numerical analysis.

Using the impurity susceptibility and specific heat coefficient, one can find Wilson's ratio in the zero temperature limit

$$W_{imp}(T_r \rightarrow 0) = \left.\frac{\gamma_{imp}}{\chi_{imp}}\right|_{T_r \rightarrow 0}. \tag{25}$$

In figure 5 we plot this value as a function of the rescaled Kondo coupling $J_r - J_{rc}$. Remember $W_{imp} = 2$ in the Kondo effect of the Fermi liquid. Here, we also obtain a similar value for the pseudogap Kondo model ($\eta_\psi = 0.001$). An important observation is that Wilson's ratio is strongly dependent on the ASL exponent. For the negative exponent Wilson's ratio becomes enhanced while it is suppressed for the positive one. This result can be understood as follows. Wilson's ratio represents the ratio of the density of states measured from specific heat and spin susceptibility. For the case of a negative exponent, bulk spinons are more coherent, thus screening an impurity spin more strongly. This enhances the density of states measured from the specific heat while it suppresses the impurity spin susceptibility owing to stronger hybridization. For the case of a positive exponent bulk spinons are less coherent, and their screening for an impurity spin becomes weak. Then, the impurity spin susceptibility becomes larger owing to weak hybridization while the density of states measured from the impurity specific heat becomes smaller. This is the reason why the impurity Wilson's ratio is larger than 2 in the case of a negative exponent while it is smaller than 2 in the case of a positive one.

The above discussion implies that Wilson's ratio can be utilized as a probe for revealing the nature of SLs and, more generally, the criticality of the bulk system. An important

issue is to determine the statistics of spinons. There are no obvious ways to determine whether they are bosonic or fermionic. Theoretically, there is no fundamental reason why one should take the bosonic or fermionic representation for the spin operator. However, the knowledge of the exponent η_ψ will be helpful in determining the nature of a possible SL phase in a frustrated antiferromagnet.

One can say, with appropriate confidence, that the sign of the exponent will be positive if gapless U(1) gauge fluctuations are absent and usual local interactions are considered. Possible bosonic SLs in frustrated antiferromagnets do not have such U(1) gauge fluctuations since non-collinear spin ordering, possibly arising in a frustrated antiferromagnet, will give rise to a gapped Z_2 SL [34]. This seems to be theoretically fundamental.

If the exponent can be measured from the impurity Wilson's ratio and its sign is negative, this excludes bosonic Z_2 SL physics and supports strong evidence for the existence of gapless gauge fluctuations, since in gapped bosonic SLs the exponent will be positive. Considering the fact that a bosonic U(1) SL can arise from a collinear antiferromagnetic order in a non-frustrated lattice such as a square one via a deconfined QCP [35, 36], one can claim that possible realization for a SL phase in the frustrated antiferromagnet is fermionic with long-range gauge interactions although there is uncertainty in determining the exponent theoretically. However, most calculations support its sign being negative, as discussed before. In this respect a fermionic U(1) SL will be a strong candidate if $\eta_\psi < 0$ is measured.

4. Summary and discussion

It is valuable to remember several assumptions for solving the ASL Kondo problem. First of all, we have considered the effects of staggered spin fluctuations on the dynamics of a magnetic impurity, ignoring those of uniform spin correlations, since antiferromagnetic spin fluctuations are most susceptible in the ASL bulk, and thus expected to give dominant contributions to this problem. In addition, ferromagnetic spin correlations do not show anomalous scaling, implying that such contributions would coincide with the pseudogap Kondo effect, thus not being so interesting. Our second assumption is in writing a spinon Green's function, where effects of gauge fluctuations are introduced in an anomalous critical exponent. In the present paper we have used the scaling exponent as a phenomenological parameter. Both assumptions are compatible since the critical exponent of the staggered spin–spin correlation function is consistent with that of the single particle propagator when both critical exponents are evaluated in a gauge invariant way.

The third assumption is rather an approximation than an assumption for solving the effective impurity action, while the above two are basic assumptions for deriving the impurity action. In the slave-boson representation of this effective impurity action we have performed the large-N_σ analysis introducing the hybridization order parameter. Although well-defined quasiparticle excitations do not exist in the case of $\eta_\psi \neq 0$, it was shown that the I-QPT occurs between the

local-moment state and the Kondo-screened phase. Evaluating the impurity spin susceptibility and specific heat coefficient, we found that Wilson's ratio depends strongly on the ASL exponent. This result has an important physical meaning because Wilson's ratio for the magnetic impurity reflects the criticality of the bulk system. This conclusion will be available to general critical systems with exact screening particularly, where the expression of the Kondo vertex is the same as that of the present paper[4]. In this respect Wilson's ratio for the magnetic impurity may be one possible probe for measuring bulk criticality.

A cautious physicist may ask how realistic the measurement of Wilson's ratio for an impurity spin in a spin liquid is. The measurement of the impurity specific heat and spin response in the ordinary Kondo effect relies on the fact that the Fermi liquid nature is well known. The impurity specific heat can be obtained by the differences for the systems with and without impurities. Obtaining the impurity spin response is very difficult in a real spin system since the background host shows a strong spin response which may dominate over the signal from the local spin, while the impurity spin response in a Fermi liquid is dominated by the local spin.

It is interesting to compare the ASL Kondo physics with the Kondo effect in a Luttinger liquid [37, 38], since the ASL can be considered as the high-dimensional realization of a Luttinger liquid. In a Luttinger liquid the Kondo interaction term can be decomposed to the forward and backward scattering channels, analogous to the uniform and staggered ones in the ASL. It was shown that the forward scattering channel is irrelevant in the renormalization group analysis up to two-loop order [38]. Similarly, in this paper we take into account only the antiferromagnetic correlation channel for the ASL Kondo effect, although it is not proven that the ferromagnetic channel is irrelevant. The backward impurity scattering in the Luttinger liquid was shown to cause anomalous scaling and, in particular, the power-law behaviour of the Kondo temperature owing to the presence of the anomalous critical exponent in the Luttinger liquid [38]. This is basically the same as the ASL Kondo effect that the ASL criticality results in anomalous scaling on the impurity physics, although there is no phase transition in the Luttinger liquid owing to its one-dimensionality.

It should be noted that it is difficult to use the present mean-field analysis to describe correct scaling behaviours in the over-screened phase since the hybridization order parameters are not regarded as dynamic variables but static ones. This approximation scheme seems to be more appropriate when quasiparticle excitations are well defined, thus the conventional Kondo screening picture is applicable. The slave-boson mean-field scheme can be improved using the non-crossing approximation (NCA) [32], where such hybridization parameters are taken to be dynamic variables, thus quantum fluctuations are more involved. Performing

the Hubbard–Stratonovich transformation for the nonlocal (for time) hopping term in equation (9), we find an effective impurity action

$$\mathcal{S}_{\text{NCA}} = \int \frac{\mathrm{d}k_0}{2\pi} \Bigg[\sum_{\sigma=1}^{N_\sigma} \chi_\sigma^\dagger (\mathrm{i}k_0 - h_\sigma + \epsilon_\chi) \chi_\sigma$$
$$+ \frac{N_\sigma}{2J_{\text{K}}} \sum_{n=1}^{N_n} b_n^{s\dagger} \gamma_0 b_n^s - \epsilon_\chi Q_\chi \Bigg]$$
$$+ \int \frac{\mathrm{d}k_0}{2\pi} \Bigg[\int \frac{\mathrm{d}k_0'}{2\pi} \frac{\Sigma_\chi(k_0) \Sigma_b(k_0')}{\mathrm{i}(k_0 - k_0') F(k_0 - k_0')}$$
$$- \sum_{\sigma=1}^{N_\sigma} \chi_\sigma^\dagger(k_0) \Sigma_b(k_0) \chi_\sigma(k_0)$$
$$+ \sum_{n=1}^{N_n} b_n^{s\dagger}(k_0) \Sigma_\chi(-k_0) b_n^s(k_0) \Bigg],$$

where $\Sigma_\chi(k_0)$ and $\Sigma_b(k_0)$ are fermion and boson self-energies, respectively, determined by the following self-consistent NCA-type equations

$$\Sigma_\chi(\tau' - \tau) = \mathcal{F}(\tau - \tau') \Bigg\langle \sum_{\sigma=1}^{N_\sigma} \chi_\sigma^\dagger(\tau) \chi_\sigma(\tau') \Bigg\rangle,$$
$$\Sigma_b(\tau - \tau') = -\mathcal{F}(\tau - \tau') \Bigg\langle \sum_{n=1}^{N_n} b_n^{s\dagger}(\tau) b_n^s(\tau') \Bigg\rangle$$

with $\mathcal{F}(k_0) = \mathrm{i}k_0 F(k_0)$. This kind of approximation is well known to catch non-Fermi liquid physics in the multi-channel Kondo model [32]. Scaling behaviours of both bosonic and fermionic self-energies are expected in the low-energy limit, causing anomalous critical physics to this system even in the case of $\eta_\psi = 0$. By inserting the expected scaling forms for both the self-energies and renormalized Green's functions in the NCA equations, we would obtain the total anomalous scaling exponents which are expected to be the sum of the critical exponents of the multi-channel pseudogap Kondo model and the ASL scaling dimension approximately, considering the presence of the ASL scaling exponent in $\mathcal{F}(\tau - \tau')$. It will be interesting to examine how the scaling exponents in the conventional bulk are affected by the presence of the ASL exponent.

Applying magnetic fields to the ASL, the impurity QPT is expected to disappear. Because external magnetic fields would result in a finite density of states at the Fermi energy, conventional Kondo physics may appear, where only the over-screened Kondo phase would occur, independent of the Kondo interaction. Furthermore, gauge fluctuations would be dissipative due to the finite density of states, and the bulk system becomes more 'Fermi liquid'-like, supporting the above expectation.

In the present analysis we did not consider scattering due to randomly distributed disorder potentials. One of the present authors has studied the role of random potentials in the ASL, and found that such a spin liquid phase remains stable against weak disorders because massless Dirac spinons at the interacting fixed point live in higher spatial dimensions than the two owing to the presence of the anomalous critical

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[4] Note that the O(3) critical bulk described by spin 1 critical fluctuations is not described by the present Kondo model. The present results cannot be applied to all critical bulk systems in spite of their generality because the vertex in the Kondo interaction differs in each system, depending on the quantum numbers of critical modes in the bulk system.

exponent [39], remember the presence of the delocalization transition above two spatial dimensions. However, it is not clear whether the diffusive nature appears or not in the ASL. If so, the presence of a finite density of states due to random potential scattering may destroy the I-QPT as in the case of magnetic fields.

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